MEASURABLE LOWER BOUNDS ON CONCURRENCE

IMAN SARGOLZAHI*, SAYYED YAHYA MIRAFZALI and MOHSEN SARBISHAEI

Department of Physics, Ferdowsi University of Mashhad
Mashhad, Iran

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We derive measurable lower bounds on concurrence of arbitrary mixed states, for both bipartite and multipartite cases. First, we construct measurable lower bonds on the *purely algebraic* bounds of concurrence [F. Mintert *et al.* (2004), Phys. Rev. lett., 92, 167902]. Then, using the fact that the sum of the square of the algebraic bounds is a lower bound of the squared concurrence, we sum over our measurable bounds to achieve a measurable lower bound on concurrence. With two typical examples, we show that our method can detect more entangled states and also can give sharper lower bonds than the similar ones.

Keywords: Measuring entanglement, Concurrence

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1. Introduction

Recently, many studies have been focused on the experimental quantification of entanglement [1]. Bell inequalities and entanglement witnesses [1, 2] can be used to detect entangled states experimentally, but they don't give any information about the amount of entanglement. In addition, quantum state tomography [3], determination of the full density operator ρ by measuring a complete set of observables, is only practical for low dimensional systems since the number of measurements needed for it grows rapidly as the dimension of the system increases. Fortunately, several methods have been introduced which let one to estimate experimentally the amount of the entanglement of an unknown ρ with no need to the full tomography [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 29, 19, 30, 20, 26, 27, 28, 21, 22, 23, 24, 25]. A simple and straightforward method is the one introduced in [8, 14, 18] for finding measurable lower bounds on an entanglement measure, namely the *concurrence* [31]. These lower bounds are in terms of the expectation values of some Hermitian operators with respect to two-fold or one-fold copy of ρ. It is worth noting that these bounds work well for weakly mixed states [32, 8, 14, 18, 5].

*sargolzahi@stu-mail.um.ac.ir, sargolzahi@gmail.com

In this paper we will use a similar procedure as [8, 14] to construct measurable lower bounds on the *purely algebraic bounds of concurrence* [33, 31]. In addition, using a theorem in Sec. II, we show that the sum of our measurable bounds leads to a measurable lower bound on the *concurrence* itself. Then, we show that this method gives better results than those introduced in [8, 14] for two typical examples.

The paper is organized as follows. In Sec. II, the concurrence and its MKB (Mintert-Kus-Buchleitner) lower bounds [33] are introduced. In Secs. III and IV, we propose measurable lower bounds on the purely algebraic bounds of concurrence [33], which are a special class of MKB bounds. The generalization to the multipartite case is given in Sec. V and we end this paper in Sec. VI with a summary and discussion.

2. Concurrence and its MKB Lower Bounds

For a pure bipartite state $|\Psi\rangle$, $|\Psi\rangle \in \mathcal{H_A} \otimes \mathcal{H_B}$, concurrence is defined as [31]:

$$C\left(\Psi\right) = \sqrt{2[\langle\Psi|\Psi\rangle^2 - tr\rho_r^2]}\,, \tag{1}$$

where ρ_r is the reduced density operator obtained by tracing over either subsystems A or B. It is obvious that iff $|\Psi\rangle$ is a product state, i.e. $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, then $C(\Psi) = 0$. Interestingly, $C(\Psi)$ can be written in terms of the expectation value of an observable with respect to two identical copies of $|\Psi\rangle$ [31, 11, 12]:

$$\begin{aligned} C\left(\Psi\right) &= \sqrt{{}_{AB}\langle\Psi|_{AB}\langle\Psi|\mathcal{A}|\Psi\rangle_{AB}|\Psi\rangle_{AB}}\,, \\ \mathcal{A} &= 4P_-^A \otimes P_-^B\,, \end{aligned} \tag{2}$$

where P_-^A (P_-^B) is the projector onto the antisymmetric subspace of $\mathcal{H_A} \otimes \mathcal{H_A}$ ($\mathcal{H_B} \otimes \mathcal{H_B}$). A possible decomposition of $\mathcal{A}$ is

$$\begin{aligned} \mathcal{A} &= \sum_\alpha |\chi_\alpha\rangle\langle\chi_\alpha|\,, \\ |\chi_\alpha\rangle &= (|xy\rangle - |yx\rangle)_A(|pq\rangle - |qp\rangle)_B\,, \end{aligned} \tag{3}$$

where $|x\rangle$ and $|y\rangle$ ($|p\rangle$ and $|q\rangle$) are two different members of an orthonormal basis of the A (B) subsystem. For mixed states the concurrence is defined as follows [31]:

$$\begin{gathered} C(\rho) = \min \sum_i p_i C(\Psi_i)\,, \\ \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|\,, \qquad p_i \geq 0\,, \qquad \sum_i p_i = 1\,, \end{gathered} \tag{4}$$

where the minimum is taken over all decompositions of ρ into pure states $|\Psi_i\rangle$. It is appropriate to write $C(\rho)$ in terms of the subnormalized states $|\psi_i\rangle$ rather than the normalized ones $|\Psi_i\rangle$:

$$\begin{gathered} C(\rho) = \min \sum_i \sqrt{\langle\psi_i|\langle\psi_i|\mathcal{A}|\psi_i\rangle|\psi_i\rangle}\,, \\ |\psi_i\rangle = \sqrt{p_i}|\Psi_i\rangle\,, \qquad \rho = \sum_i |\psi_i\rangle\langle\psi_i|\,; \end{gathered} \tag{5}$$

since all decompositions of ρ into subnormalized states are related to each other by unitary matrices [3]: consider an arbitrary decomposition of $\rho = \sum_j |\varphi_j\rangle\langle\varphi_j|$ (As a special case,

one can choose $|\varphi_j\rangle = \sqrt{\lambda_j}|\Phi_j\rangle$, where $|\Phi_j\rangle$ and λ_j are eigenvectors and eigenvalues of ρ respectively: $\rho = \sum_j \lambda_j |\Phi_j\rangle\langle\Phi_j|$.), for any other decomposition of $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$ we have [3]:

$$|\psi_i\rangle = \sum_j U_{ij}|\varphi_j\rangle\,, \qquad \sum_i U^\dagger_{ki}U_{ij} = \delta_{jk}\,. \tag{6}$$

So Eq. (5) can be written as:

$$\begin{aligned} C(\rho) &= \min_U \sum_i \sqrt{\sum_{jklm} U_{ij}U_{ik}\mathcal{A}^{lm}_{jk}U^\dagger_{li}U^\dagger_{mi}}\,, \\ \mathcal{A}^{lm}_{jk} &= \langle\varphi_l|\langle\varphi_m|\mathcal{A}|\varphi_j\rangle|\varphi_k\rangle\,. \end{aligned} \tag{7}$$

From the definition of $C(\rho)$ in Eq. (4) it is obvious that $C(\rho) = 0$ iff ρ can be decomposed into product states. In other words, $C(\rho) = 0$ iff ρ is separable. In addition, it can be shown that the concurrence is an entanglement monotone [34] (An entanglement monotone is a function of ρ which does not increase, on average, under LOCC and vanishes for separable states [35].). But, except for the two-qubit case [36], $C(\rho)$ can not be computed in general; i.e., in general, one can not find the U which minimizes Eq. (7). Any numerical method for finding the U which minimizes Eq. (7) leads to an upper bound for $C(\rho)$. So, finding lower bounds on $C(\rho)$ is desirable. So far, several lower bounds for $C(\rho)$ have been introduced [33, 31, 37, 38, 39, 40, 41, 42, 43, 5, 8, 13, 14, 18, 19, 21, 22, 23, 24]. One of them is that introduced by F. Mintert *et al.* in [33, 31]. Now, we redrive their lower bounds in a slightly different form to make them more suitable for finding measurable lower bounds in the following sections.

Assume that the decomposition of ρ which minimizes Eq. (5) is $\rho = \sum_j |\xi_j\rangle\langle\xi_j|$, then from Eqs. (3) and (5), we have:

$$C(\rho) = \sum_j \sqrt{\sum_\alpha |\langle\chi_\alpha|\xi_j\rangle|\xi_j\rangle|^2} \geq \sum_j |\langle\chi_\beta|\xi_j\rangle|\xi_j\rangle| \geq \min_{\{|\psi_i\rangle\}} \sum_i |\langle\chi_\beta|\psi_i\rangle|\psi_i\rangle|\,, \tag{8}$$

where $|\chi_\beta\rangle \in \{|\chi_\alpha\rangle\}$, and the minimum is taken over all decompositions of ρ as $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$. Now, using Eq. (6), we have:

$$\begin{aligned} \min_{\{|\psi_i\rangle\}} \sum_i |\langle\chi_\beta|\psi_i\rangle|\psi_i\rangle| &= \min_U \sum_i |\sum_{jk} U_{ij}T^\beta_{jk}U^\top_{ki}| = \min_U \sum_i |\left[UT^\beta U^\top\right]_{ii}|\,, \\ T^\beta_{jk} &= \langle\chi_\beta|\varphi_j\rangle|\varphi_k\rangle\,. \end{aligned} \tag{9}$$

Since T^β is a symmetric matrix, the minimum in Eq. (9) can be computed and we have [31]:

$$\min_U \sum_i |\left[UT^\beta U^\top\right]_{ii}| = \max\{0, S^\beta_1 - \sum_{l>1} S^\beta_l\}\,, \tag{10}$$

where S^β_l are the singular values of T^β, in decreasing order. The above expression is what was named *purely algebraic lower bound* of concurrence in [31, 33] and we will refer to it as $ALB(\rho)$.

Let us define

$$|\tau\rangle = \sum_\alpha z^*_\alpha|\chi_\alpha\rangle\,, \qquad \sum_\alpha |z_\alpha|^2 = 1\,. \tag{11}$$

Obviously, $|\tau\rangle$ is an element of another (normalized to 2) basis of $P_-^A \otimes P_-^B$, $\{|\chi'_\alpha\rangle\}$. Then:

$$\begin{aligned} |\tau\rangle &\equiv |\chi'_1\rangle \,, \\ \mathcal{A} = \sum_\alpha |\chi_\alpha\rangle\langle\chi_\alpha| &= |\tau\rangle\langle\tau| + \sum_{\alpha>1} |\chi'_\alpha\rangle\langle\chi'_\alpha| \,. \end{aligned} \tag{12}$$

Again, as the inequality (8), we have:

$$\begin{aligned} C(\rho) &= \sum_j \sqrt{\sum_\alpha |\langle\chi'_\alpha|\xi_j\rangle|\xi_j\rangle|^2} \geq \sum_j |\langle\tau|\xi_j\rangle|\xi_j\rangle| \\ &\geq \min_{\{|\psi_i\rangle\}} \sum_i |\langle\tau|\psi_i\rangle|\psi_i\rangle| \\ &= \min_U \sum_i | \left[U\mathcal{T}U^\top\right]_{ii} | = \max\{0, S_1^\tau - \sum_{l>1} S_l^\tau\} \,, \\ \mathcal{T}_{jk} &= \langle\tau|\varphi_j\rangle|\varphi_k\rangle = \sum_\alpha z_\alpha T_{jk}^\alpha \,, \end{aligned} \tag{13}$$

where S_l^τ are the singular values of $\mathcal{T}$, in decreasing order. The above expression is the general form of the lower bounds introduced in [33, 31] and we call it $LB(\rho)$.

We end this section by proving a useful theorem: if $\{|\chi'_\alpha\rangle\}$ be an orthogonal (normalized to 2) basis of $P_-^A \otimes P_-^B$, i.e. $\mathcal{A} = \sum_\alpha |\chi'_\alpha\rangle\langle\chi'_\alpha|$, then:

$$\begin{aligned} C^2(\rho) &= \sum_{ij} \sqrt{\sum_\alpha |\langle\chi'_\alpha|\xi_i\rangle|\xi_i\rangle|^2} \sqrt{\sum_\alpha |\langle\chi'_\alpha|\xi_j\rangle|\xi_j\rangle|^2} \\ &\geq \sum_{ij} \sum_\alpha |\langle\chi'_\alpha|\xi_i\rangle|\xi_i\rangle||\langle\chi'_\alpha|\xi_j\rangle|\xi_j\rangle| \\ &= \sum_\alpha \left(\sum_i |\langle\chi'_\alpha|\xi_i\rangle|\xi_i\rangle|\right)^2 \geq \sum_\alpha \left[LB_\alpha(\rho)\right]^2 \,, \\ LB_\alpha(\rho) &= \min_{\{|\psi_i\rangle\}} \sum_i |\langle\chi'_\alpha|\psi_i\rangle|\psi_i\rangle| \,. \end{aligned} \tag{14}$$

In proving the above theorem we have used the Cauchy-Schwarz inequality. Obviously, any entangled ρ which can not be detected by LB_α, can not be detected by $\sum_\alpha \left[LB_\alpha(\rho)\right]^2$ either; i.e., $\sum_\alpha \left[LB_\alpha(\rho)\right]^2$ is not a more powerful criteria than LB_α, but, quantitatively, it may lead to a better lower bound for $C(\rho)$.

It should be mentioned that the above theorem is, in fact, the generalization of what has been proved in [42]. There, it was shown that:

$$\begin{aligned} \tau(\rho) &= \sum C_{mn}^2(\rho) \leq C^2(\rho) \,, \\ C_{mn}(\rho) &= \min_{\{|\psi_i\rangle\}} \sum_i |\langle\psi_i|L_{m_A} \otimes L_{n_B}|\psi_i^*\rangle| \,, \end{aligned} \tag{15}$$

where L_{m_A} and L_{n_B} are generators of $SO(d_A)$ and $SO(d_B)$ respectively ($d_{A/B} = dim(\mathcal{H}_{\mathcal{A}/\mathcal{B}})$), and $|\psi_i^*\rangle$ is the complex conjugate of $|\psi_i\rangle$ in the computational basis. In this basis L_{m_A} and L_{n_B} are [44]:

$$L_{m_A} = |x\rangle_A\langle y| - |y\rangle_A\langle x| \,, \qquad L_{m_B} = |p\rangle_B\langle q| - |q\rangle_B\langle p| \,.$$

For an arbitrary $|\psi\rangle$, according to the definition of $|\chi_\alpha\rangle$ in Eq. (3), it can be seen that:

$$|\langle\psi|L_{m_A}\otimes L_{n_B}|\psi^*\rangle| = |\langle\chi_\alpha|\psi\rangle|\psi\rangle|\,. \tag{16}$$

So:

$$\begin{aligned} C_{mn}(\rho) &= ALB_\alpha(\rho)\,, \\ ALB_\alpha(\rho) &= \min_{\{|\psi_i\rangle\}}\sum_i |\langle\chi_\alpha|\psi_i\rangle|\psi_i\rangle|\,. \end{aligned} \tag{17}$$

So what was proved in [42] is, in fact, the special case of $|\chi'_\alpha\rangle = |\chi_\alpha\rangle$ in expression (14). In addition, since ALB_α can detect bound entangled states [33, 31], this claim of [42] that any state for which $\tau(\rho) > 0$ is distillable, is not correct.

3. Measurable Lower Bounds in terms of Two Identical Copies of ρ

As we have seen in Eq. (2) the concurrence of a pure state $|\Psi\rangle$ can be written in terms of the expectation value of the observable $\mathcal{A}$ with respect to two identical copies of $|\Psi\rangle$. For an arbitrary mixed state ρ_{AB}, it has been shown that [8]:

$$\begin{aligned} C^2(\rho_{AB}) \geq tr\left(\rho_{AB}\otimes\rho_{AB}V_{(i)}\right)\,, \qquad i=1,2\,; \\ V_{(1)} = 4\left(P_-^A - P_+^A\right)\otimes P_-^B\,, \qquad\qquad V_{(2)} = 4P_-^A\otimes\left(P_-^B - P_+^B\right)\,, \end{aligned} \tag{18}$$

where P_+^A (P_+^B) is the projector onto the symmetric subspace of $\mathcal{H_A}\otimes\mathcal{H_A}$ ($\mathcal{H_B}\otimes\mathcal{H_B}$). The above expression means that measuring $V_{(i)}$ on two identical copies of ρ, i.e. $\rho\otimes\rho$, gives us a measurable *lower* bound on $C^2(\rho)$. It is worth noting that if the entanglement of ρ can be detected by $V_{(i)}$, then ρ is distillable [24].

As one can see from expression (13), the LB of a pure state $|\Psi\rangle$ can also be written in terms of the expectation value of the observable $|\tau\rangle\langle\tau|$ with respect to two identical copies of $|\Psi\rangle$. Now, for an arbitrary mixed state ρ, can we find an observable V such that the following inequality holds?

$$LB^2(\rho) \geq tr\left(\rho\otimes\rho V\right)\,, \tag{19}$$

Fortunately for the special case of $|\tau\rangle = |\chi_\alpha\rangle$, where $|\chi_\alpha\rangle$ are defined in Eq. (3), we can do so.

Assume that the decomposition of ρ which gives the minimum in Eq. (9) is $\rho = \sum_i |\theta_i^\alpha\rangle\langle\theta_i^\alpha|$, i.e.:

$$ALB_\alpha(\rho) = \sum_i |\langle\chi_\alpha|\theta_i^\alpha\rangle|\theta_i^\alpha\rangle|\,. \tag{20}$$

In addition, assume that for a Hermitian operator V_α, which acts on $\mathcal{H_A}\otimes\mathcal{H_B}\otimes\mathcal{H_A}\otimes\mathcal{H_B}$, and arbitrary $|\psi\rangle$ and $|\varphi\rangle$, $|\psi\rangle\in\mathcal{H_A}\otimes\mathcal{H_B}$ and $|\varphi\rangle\in\mathcal{H_A}\otimes\mathcal{H_B}$, we have:

$$|\langle\chi_\alpha|\psi\rangle|\psi\rangle||\langle\chi_\alpha|\varphi\rangle|\varphi\rangle| \geq \langle\psi|\langle\varphi|V_\alpha|\psi\rangle|\varphi\rangle\,. \tag{21}$$

Now, from the expressions (20) and (21), we have:

$$ALB_\alpha^2(\rho) = \sum_{ij}|\langle\chi_\alpha|\theta_i^\alpha\rangle|\theta_i^\alpha\rangle||\langle\chi_\alpha|\theta_j^\alpha\rangle|\theta_j^\alpha\rangle| \geq \sum_{ij}\langle\theta_i^\alpha|\langle\theta_j^\alpha|V_\alpha|\theta_i^\alpha\rangle|\theta_j^\alpha\rangle = tr\left(\rho\otimes\rho V_\alpha\right)\,. \tag{22}$$

So, for any V_α satisfying inequality (21), measuring V_α on two identical copies of ρ gives a lower bound on $ALB^2_\alpha(\rho)$. We can prove that the inequality (21) holds for(see the Appendix):

$$\begin{array}{cc} V_\alpha = V_{(1)\alpha} = \mathcal{M}V_{(1)}\mathcal{M}\,, & V_\alpha = V_{(2)\alpha} = \mathcal{M}V_{(2)}\mathcal{M}\,, \\ \multicolumn{2}{c}{\mathcal{M} = \mathcal{M}_A \otimes \mathcal{M}_A \otimes \mathcal{M}_B \otimes \mathcal{M}_B\,,} \\ \mathcal{M}_A = |x\rangle\langle x| + |y\rangle\langle y|\,, & \mathcal{M}_B = |p\rangle\langle p| + |q\rangle\langle q|\,, \end{array} \tag{23}$$

where $|x\rangle$, $|y\rangle$, $|p\rangle$, $|q\rangle$ are introduced in Eq. (3) (note that $|\chi_\alpha\rangle\langle\chi_\alpha| = \mathcal{MAM}$). In addition, for any V_α such as

$$V_\alpha = c_1 V_{(1)\alpha} + c_2 V_{(2)\alpha}\,, \qquad c_1 \geq 0\,, \qquad c_2 \geq 0\,, \qquad c_1 + c_2 = 1\,, \tag{24}$$

inequalities (21) and, consequently, (22) also hold.

According to the definition of V_α in Eqs. (23) and (24), we have:

$$\begin{array}{c} tr\,(\rho \otimes \rho V_\alpha) = tr\,(\varrho \otimes \varrho V_\alpha)\,, \\ \varrho = \mathcal{M}_A \otimes \mathcal{M}_B \rho \mathcal{M}_A \otimes \mathcal{M}_B\,, \end{array} \tag{25}$$

which means that if V_α detects the entanglement of ρ, it has, in fact, detected the entanglement of a two-qubit submatrix of ρ. Any ρ which has an entangled two-qubit submatrix is distillable [45]. So any ρ which is detected by V_α is distillable.

The right hand side of the inequality (18) is invariant under local unitary transformations [8]:

$$\begin{array}{c} tr\left(\rho \otimes \rho V_{(i)}\right) = tr\left(\rho' \otimes \rho' V_{(i)}\right)\,, \\ \rho' = U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger\,, \end{array} \tag{26}$$

where U_A and U_B are arbitrary unitary operators. This is so because $U_A^\dagger \otimes U_A^\dagger P_\pm^A U_A \otimes U_A = P_\pm^A$ and $U_B^\dagger \otimes U_B^\dagger P_\pm^B U_B \otimes U_B = P_\pm^B$. So, the choices of local bases in the definition of $V_{(i)}$ in (18) are not important since all the choices lead to the same result. But, according to the definition of V_α in Eqs. (23) and (24), the right hand side of the inequality (22) is not invariant under local unitary transformations. It is however expected since the $ALB_\alpha(\rho)$ is not invariant under such transformations either.

Using Eqs. (23) and (24), it can be shown simply that the right hand side of the inequality (22) is invariant under the following transformations:

$$\begin{array}{cc} \multicolumn{2}{c}{tr\,(\rho \otimes \rho V_\alpha) = tr\,(\rho' \otimes \rho' V_\alpha)\,,} \\ \multicolumn{2}{c}{\rho' = u_A \otimes u_B \rho u_A^\dagger \otimes u_B^\dagger\,,} \\ \mathcal{M}_A u_A \mathcal{M}_A = u_A, & u_A u_A^\dagger = u_A^\dagger u_A = \mathcal{M}_A, \\ \mathcal{M}_B u_B \mathcal{M}_B = u_B, & u_B u_B^\dagger = u_B^\dagger u_B = \mathcal{M}_B, \\ \multicolumn{2}{c}{\Rightarrow tr(\rho') \leq 1.} \end{array} \tag{27}$$

$|\chi_\alpha\rangle$ is also invariant, up to a phase, under the above transformations, i.e. $u_A \otimes u_A \otimes u_B \otimes u_B|\chi_\alpha\rangle = e^{i\beta}|\chi_\alpha\rangle$ and $0 \leq \beta \leq 2\pi$, but it is not so for the $ALB_\alpha(\rho)$. Consider the decomposition of ρ into pure states as $\rho = \sum_i |\theta_i^\alpha\rangle\langle\theta_i^\alpha|$. From Eq. (27) we know that there is a decomposition of ρ' into pure states as $\rho' = \sum_i |\theta_i'^\alpha\rangle\langle\theta_i'^\alpha|$, where $|\theta_i'^\alpha\rangle = u_A \otimes u_B|\theta_i^\alpha\rangle$. So, using Eq. (20):

$$\sum_i |\langle\chi_\alpha|\theta_i'^\alpha\rangle|\theta_i'^\alpha\rangle| = \sum_i |\langle\chi_\alpha|\theta_i^\alpha\rangle|\theta_i^\alpha\rangle| = ALB_\alpha(\rho). \tag{28}$$

But

$$\sum_i |\langle\chi_\alpha|\theta_i^{'\alpha}\rangle|\theta_i^{'\alpha}\rangle| \geq \min_{\{|\psi_j'\rangle\}} \sum_j |\langle\chi_\alpha|\psi_j'\rangle|\psi_j'\rangle| = ALB_\alpha(\rho'), \tag{29}$$

where the minimum is taken over all decompositions of ρ' into pure states: $\rho' = \sum_j |\psi_j'\rangle\langle\psi_j'|$. So:

$$ALB_\alpha(\rho') \leq ALB_\alpha(\rho). \tag{30}$$

Note that expressions (22), (27) and (30) show that $tr(\rho \otimes \rho V_\alpha)$ bounds the amount of $ALB_\alpha^2(\rho')$, for all possible ρ' in Eq. (27), from below.

Now, using inequalities (14) and (22):

$$C^2(\rho) \geq \sum_\alpha ALB_\alpha^2(\rho) \geq \sum_\alpha tr\left(\rho \otimes \rho V_\alpha\right), \tag{31}$$

where the summation is only over those α for which $tr\left(\rho \otimes \rho V_\alpha\right) \geq 0$.

Example 1. In a $d \times d$ dimensional Hilbert space, isotropic states are defined as [2]:

$$\begin{aligned} \rho_{_F} &= \frac{1-F}{d^2-1}\left(I - |\phi^+\rangle\langle\phi^+|\right) + F|\phi^+\rangle\langle\phi^+|, \\ |\phi^+\rangle &= \sum_{i=1}^d \frac{1}{\sqrt{d}}|i_A i_B\rangle, \\ 0 \leq F \leq 1, &\qquad F = \langle\phi^+|\rho_{_F}|\phi^+\rangle. \end{aligned} \tag{32}$$

The concurrence of $\rho_{_F}$ is known and we have [34]:

$$C\left(\rho_{_F}\right) = max\left\{0, \sqrt{\frac{2d}{d-1}}\left(F - \frac{1}{d}\right)\right\}. \tag{33}$$

If we rewrite $\rho_{_F}$ as

$$\rho_{_F} = \frac{1-F}{d^2-1}I + \frac{Fd^2-1}{d^2-1}|\phi^+\rangle\langle\phi^+| \equiv gI + h|\phi^+\rangle\langle\phi^+|,$$

then:

$$tr\left(\rho_{_F} \otimes \rho_{_F} V_{(i)}\right) = 2d\left(d-1\right)\left[\frac{h^2}{d^2} - dg^2 - \frac{2}{d}gh\right]. \tag{34}$$

In Eq. (23), if we choose $\{x = p, y = q\}$, then:

$$tr\left(\rho_{_F} \otimes \rho_{_F} V_\alpha\right) = 4\left[\frac{h^2}{d^2} - 2g^2 - \frac{2}{d}gh\right],$$

and the expectation values of other V_α are not positive. Since the case $\{x = p, y = q\}$ occurs $n = \frac{d(d-1)}{2}$ times in a $d \times d$ dimensional system, we have:

$$tr\left(\rho_{_F} \otimes \rho_{_F} \sum_\alpha V_\alpha\right) = 2d(d-1)\left[\frac{h^2}{d^2} - 2g^2 - \frac{2}{d}gh\right], \tag{35}$$

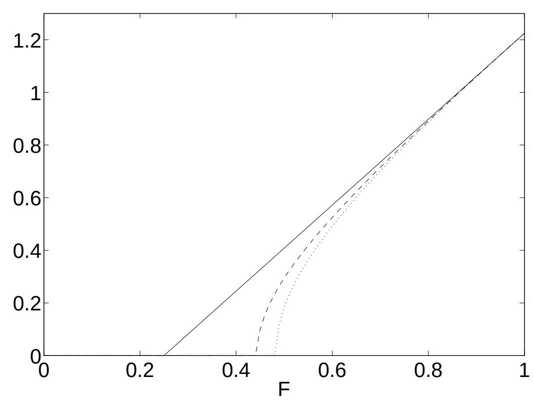

Fig. 1. Comparison of Eqs. (34), dotted line, and (35), dashed line, for $d = 4$. The solid line is the exact value of concurrence, Eq. (33). The lower bounds given by $V_{(i)}$ and $\sum_\alpha V_\alpha$ are set to zero when the right hand sides of Eqs. (34) and (35) are less than zero.

where the summation is only over those V_α for which $\{x = p, y = q\}$. For $d > 2$, Eq. (35) gives a better result than Eq. (34) (Fig. 1). For $d = 2$ both give the same result, as we expect from Eq. (23).

4. Measurable Lower Bounds in terms of One Copy of ρ

From the experimental point of view, any lower bound which is defined in terms of the expectation value of an observable with respect to two identical copies of ρ, encounters, at least, two problems. First, for measuring $V_{(i)}$ or V_α we need to measure in an entangled basis in both parts A and B. Measuring in an entangled basis is more difficult than measuring in a separable one [12]. Second, it is not clear that the state which enters the measuring devices is really as $\rho \otimes \rho$ even if we can produce such state at the source place [46, 10]. So, having lower bounds in terms of the expectation value of an observable with respect to *one* copy of ρ is more desirable.

Using:

$$\begin{aligned} C(\rho)C(\sigma) &\geq tr\left(\rho \otimes \sigma V_{(i)}\right) , \qquad i = 1, 2\,; \\ \Rightarrow C(\rho) &\geq \frac{1}{C(\sigma)} tr\left(\rho \otimes \sigma V_{(i)}\right) , \end{aligned} \tag{36}$$

for arbitrary ρ and σ, F. Mintert has introduced the following measurable lower bound on $C(\rho)$ [14]:

$$C(\rho) \geq -tr\left(\rho W_\sigma\right) , \qquad W_\sigma = \frac{-1}{C(\sigma)} tr_2\left(I \otimes \sigma V_{(i)}\right) , \tag{37}$$

where σ is a pre-determined entangled state and the partial trace is taken over the second copy of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. If $C(\sigma)$ is not computable, which is the case for almost all mixed σ, an upper bound of $C(\sigma)$ can be used in the definition of W_σ. From inequality (37), it is obvious

that for any separable state: $tr\,(\rho_s W_\sigma) \geq 0$. If, at least, for one entangled state $tr\,(\rho_e W_\sigma) < 0$, then W_σ is an entanglement witness [2].

We can, also, construct measurable lower bounds in terms of one copy of ρ by using inequality (21). Suppose that the decomposition of σ which gives the minimum in Eq. (9) is $\sigma = \sum_j |\gamma_j^\alpha\rangle\langle\gamma_j^\alpha|$, i.e.:

$$ALB_\alpha(\sigma) = \sum_j |\langle\chi_\alpha|\gamma_j^\alpha\rangle|\gamma_j^\alpha\rangle| \, . \tag{38}$$

Using expressions (20), (21) and (38):

$$\begin{aligned} [ALB_\alpha(\rho)]\,[ALB_\alpha(\sigma)] &= \sum_{ij} |\langle\chi_\alpha|\theta_i^\alpha\rangle|\theta_i^\alpha\rangle||\langle\chi_\alpha|\gamma_j^\alpha\rangle|\gamma_j^\alpha\rangle| \\ &\geq \sum_{ij} \langle\theta_i^\alpha|\langle\gamma_j^\alpha|V_\alpha|\theta_i^\alpha\rangle|\gamma_j^\alpha\rangle = -tr\,(\rho W'_{\sigma\alpha}) \ , \\ W'_{\sigma\alpha} &= -tr_2\,(I \otimes \sigma V_\alpha) \ . \end{aligned}$$

So:

$$ALB_\alpha(\rho) \geq -tr\,(\rho W_{\sigma\alpha}) \ , \qquad\qquad W_{\sigma\alpha} = \frac{1}{ALB_\alpha(\sigma)} W'_{\sigma\alpha} \, , \tag{39}$$

where σ is a pre-determined entangled state for which $ALB_\alpha(\sigma) > 0$. Note that, in contrast to $C(\sigma)$, $ALB_\alpha(\sigma)$ is always computable, so we never need to use an upper bound of it in the definition of $W_{\sigma\alpha}$. In addition, it can be shown simply that

$$tr\,(\rho W_{\sigma\alpha}) = tr\,(\varrho W_{\sigma\alpha}) \ , \tag{40}$$

where ϱ is defined in Eq. (25). So any ρ which is detected by $W_{\sigma\alpha}$ is distillable. Also, using inequalities (14) and (39):

$$C^2(\rho) \geq \sum_\alpha [ALB_\alpha(\rho)]^2 \geq \sum_\alpha [tr\,(\rho W_{\sigma\alpha})]^2 \ , \tag{41}$$

where the summation is over those α for which $tr\,(\rho W_{\sigma\alpha}) \leq 0$.

For isotropic states,using expressions (37) or (41) (by choosing $\sigma = |\phi^+\rangle\langle\phi^+|$) gives the exact value of $C(\rho_{_F})$ for arbitrary d. In the following, we give an example for which the expression (41) gives better results than the expression (37).

Example 2. Consider a two-qutrit system which is initially in the pure state

$$|\Phi\rangle = \sqrt{\lambda_0}|01\rangle + \sqrt{\lambda_1}|12\rangle + \sqrt{\lambda_2}|20\rangle \, , \tag{42}$$

and its time evolution is given by the following Master equation [14]:

$$\begin{aligned} \dot{\rho} &= \mathcal{L}\rho \, , \\ \mathcal{L} &= \mathcal{L}_A \otimes 1_B + 1_A \otimes \mathcal{L}_B \, , \end{aligned} \tag{43}$$

where $\mathcal{L}_{A/B}$, for a one-qutrit $\rho_{A/B}$, is

$$\mathcal{L}_{A/B} = \frac{\Gamma}{2}\left(2\gamma\rho_{A/B}\gamma^\dagger - \rho_{A/B}\gamma^\dagger\gamma - \gamma^\dagger\gamma\rho_{A/B}\right) \ ,$$

and γ is the coupling matrix for the spontaneous decay:

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

To construct $W_{\sigma\alpha}$ in expression (39) and W_σ in expression (37), we choose

$$\begin{aligned} \sigma &= |\Phi_{ME}\rangle\langle\Phi_{ME}| \,, \\ |\Phi_{ME}\rangle &= \frac{1}{\sqrt{3}} \left(|01\rangle + |12\rangle + |20\rangle\right) . \end{aligned} \tag{44}$$

It can be shown simply that for three $|\chi_\alpha\rangle$, for which $\{p = x \oplus 1, q = y \oplus 1\}$ ($\oplus$ is the sum modulo 3), $ALB_\alpha(\sigma) = 2/3$, and $ALB_\alpha(\sigma) = 0$ for other $|\chi_\alpha\rangle$. So, using expression (39), we can construct three $W_{\sigma\alpha}$ as ($x = 0, 1, 2$ and $y = x \oplus 1$):

$$\begin{aligned} W_{\sigma\alpha} &= |x, y\oplus 1\rangle\langle x, y\oplus 1| + |y, x\oplus 1\rangle\langle y, x\oplus 1| - |x, x\oplus 1\rangle\langle y, y\oplus 1| - |y, y\oplus 1\rangle\langle x, x\oplus 1| \\ &= |x, y\oplus 1\rangle\langle x, y\oplus 1| + |y, x\oplus 1\rangle\langle y, x\oplus 1| - \frac{1}{2}\left(\sigma_1^{xy} \otimes \sigma_1^{x\oplus 1, y\oplus 1} - \sigma_2^{xy} \otimes \sigma_2^{x\oplus 1, y\oplus 1}\right) , \\ \sigma_1^{ab} &= |a\rangle\langle b| + |b\rangle\langle a| \,, \qquad \sigma_2^{ab} = -i\left(|a\rangle\langle b| - |b\rangle\langle a|\right) . \end{aligned} \tag{45}$$

Also, using expression (37), we can show that:

$$W_\sigma = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} W_{\sigma\alpha} \,. \tag{46}$$

As we can see from Eqs. (45) and (46), the number of local observables needed for measuring W_σ or three $W_{\sigma\alpha}$ is the same and is equal to 12, which is less than what is needed for a full tomography. Also, note that $\{|l, m \oplus 1\rangle\}$ is an orthonormal basis of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. So, at least from the theoretical point of view, all the observables $|l, m\oplus 1\rangle\langle l, m\oplus 1|$ can be measured using only one set up. In such cases, for measuring W_σ or three $W_{\sigma\alpha}$, we only need 7 different set up of local measurements. The comparison of the results of inequalities (37) and (41), for two typical $\{\lambda_i\}$, is given in Fig. 2.

5. Extending to Multipartite Systems

In a bipartite system, any Hermitian operator which, for arbitrary $|\psi\rangle$ and $|\varphi\rangle$, satisfies the inequality

$$C(\psi)C(\varphi) \geq \langle\psi|\langle\varphi|V|\psi\rangle|\varphi\rangle \,, \tag{47}$$

gives a measurable lower bound on $C^2(\rho)$, i.e. $C^2(\rho) \geq tr\,(\rho \otimes \rho V)$ [10]. This can be proved simply by writing ρ in terms of its extremal decomposition $\rho = \sum_j |\xi_j\rangle\langle\xi_j|$. In [18] it was shown how to use such V to construct measurable lower bounds for multipartite concurrence. Following a similar procedure, we construct measurable lower bounds on multipartite concurrence using V_α. As the previous sections, we will use the inequality (21) instead of the inequality (47). In other words, we will work with the algebraic lower bounds of $C(\rho)$ rather than the concurrence itself.

The concurrence of an N-partite pure state $|\Psi\rangle$, $|\Psi\rangle \in \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N}$, is defined as [31]:

$$C(\Psi) = 2^{1-\frac{N}{2}} \sqrt{\sum_l C_l^2(\Psi)} \,, \tag{48}$$

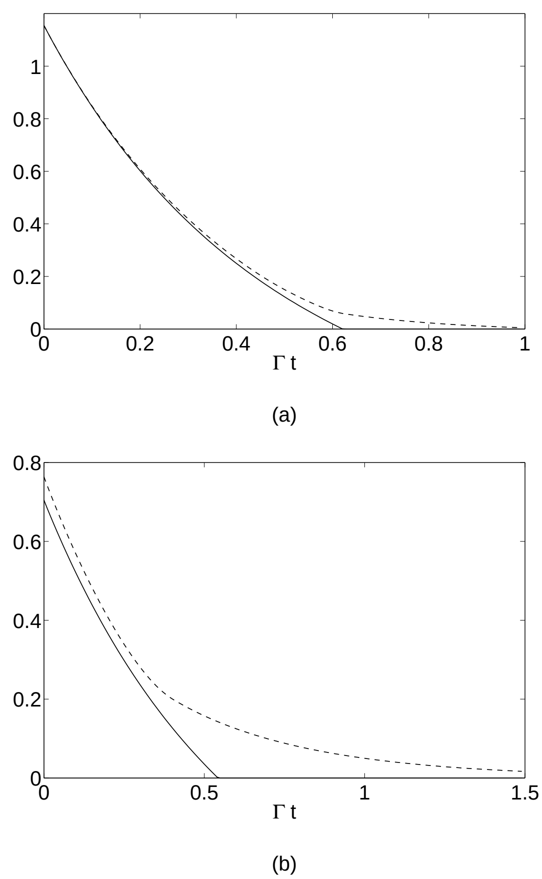

Fig. 2. Comparing the lower bounds given by (37), solid line, and (41), dashed line, for two typical $\{\lambda_i\}$: a) $\lambda_i = 1/3$; b) $\{\lambda_0 = 1/12, \lambda_1 = 5/6, \lambda_2 = 1/12\}$. When the lower bound given by W_σ is less than zero, we set it to zero.

where $\sum_l$ is the summation over all possible subdivisions of $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N}$ into two subsystems, and C_l is the related bipartite concurrence. For example, for a 3-partite system we have three C_l, namely $C_{1,23}, C_{12,3}$ and $C_{13,2}$. As before we have:

$$C_l^2\,(\Psi) = \langle\Psi|\langle\Psi|\mathcal{A}_l|\Psi\rangle|\Psi\rangle\,, \qquad \mathcal{A}_l = \sum_{\alpha_l} |\chi_{\alpha_l}\rangle\langle\chi_{\alpha_l}|\,, \tag{49}$$

where $|\chi_{\alpha_l}\rangle$ are the same as $|\chi_\alpha\rangle$ which have been defined in Eq. (3). Obviously, they are constructed according to the related subdivision denoted by l. So:

$$C(\Psi) = 2^{1-\frac{N}{2}} \sqrt{\sum_{l,\alpha_l} |\langle\chi_{\alpha_l}|\Psi\rangle|\Psi\rangle|^2} = 2^{1-\frac{N}{2}} \sqrt{\sum_{\gamma} |\langle\chi_{\gamma}|\Psi\rangle|\Psi\rangle|^2}\,, \tag{50}$$

where instead of l and α_l we have used a collective index γ. From now on, everything is as the bipartite case, except that we deal with the summation over γ instead of α. The definition of concurrence for mixed states is as follows:

$$\begin{aligned} C(\rho) = \min_{\{|\psi_i\rangle\}} \sum_i C(\psi_i) = \min_{\{|\psi_i\rangle\}} \sum_i 2^{1-\frac{N}{2}} \sqrt{\langle\psi_i|\langle\psi_i|\mathcal{A}'|\psi_i\rangle|\psi_i\rangle}\,, \\ \mathcal{A}' = \sum_\gamma |\chi_\gamma\rangle\langle\chi_\gamma|\,, \end{aligned} \tag{51}$$

where the minimization is over all decompositions of ρ into subnormalized states $|\psi_i\rangle$: $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$. It is worth noting that $C(\rho)$, as difined in Eq. (51), is an entanglement monotone for the multipartite case too [47].

If we define $|\chi'_v\rangle = \sum_\gamma U'_{v\gamma}|\chi_\gamma\rangle$, where U' is a unitary matrix, then $\mathcal{A}' = \sum_\gamma |\chi_\gamma\rangle\langle\chi_\gamma| = \sum_\gamma |\chi'_\gamma\rangle\langle\chi'_\gamma|$. So, by similar reasoning leading to inequality (13), we have:

$$\begin{aligned} C(\rho) \geq LB_\tau(\rho) = \min_{\{|\psi_i\rangle\}} \sum_i 2^{1-\frac{N}{2}} |\langle\tau|\psi_i\rangle|\psi_i\rangle|\,, \\ |\tau\rangle \equiv |\chi'_1\rangle = \sum_\gamma z_\gamma^* |\chi_\gamma\rangle\,, \qquad \sum_\gamma |z_\gamma|^2 = 1\,. \end{aligned} \tag{52}$$

As before, in contrast to $C(\rho)$, $LB_\tau(\rho)$ is always computable. We also have:

$$C^2(\rho) \geq \sum_\gamma \left[LB_\gamma(\rho)\right]^2\,, \qquad LB_\gamma(\rho) = \min_{\{|\psi_i\rangle\}} 2^{1-\frac{N}{2}} \sum_\gamma |\langle\chi'_\gamma|\psi_i\rangle|\psi_i\rangle|\,. \tag{53}$$

The above expression is the counterpart of the inequality (14) for the multipartite case. What was proved in [43], neglecting an unimportant constant in the definition of $C(\rho)$, is, in fact, the inequality (53) for the special case of $|\chi'_\gamma\rangle = |\chi_\gamma\rangle$ (see Eqs. (16) and (17)).

According to the inequality (21),for any $|\chi_\gamma\rangle$:

$$|\langle\chi_\gamma|\psi\rangle|\psi\rangle||\langle\chi_\gamma|\varphi\rangle|\varphi\rangle| \geq \langle\psi|\langle\varphi|V_\gamma|\psi\rangle|\varphi\rangle\,, \tag{54}$$

where V_γ are the same as V_α introduced in Eqs. (23) and (24), defined according to the related $|\chi_\gamma\rangle$. So:

$$C^2(\rho) \geq 2^{2-N} \sum_\gamma tr\left(\rho \otimes \rho V_\gamma\right)\,, \tag{55}$$

where the summation is over those γ for which $tr\,(\rho\otimes\rho V_\gamma)\geq 0$. Also, we have:

$$C^2(\rho)\geq\sum_\gamma\left[tr\,(\rho W_{\sigma\gamma})\right]^2\,,\qquad W_{\sigma\gamma}=\frac{-2^{2-N}}{ALB_\gamma(\sigma)}tr_2\,(I\otimes\sigma V_\gamma)\,, \tag{56}$$

where σ is a pre-determined density operator with $ALB_\gamma(\sigma)>0$, and the summation is over those γ for which $tr\,(\rho W_{\sigma\gamma})\leq 0$.

6. Summery and Discussion

Inequality (21) is the main relation of this paper. Using this expression, we have constructed measurable lower bounds on concurrence in term of both one copy or two identical copies of ρ. We have proved that the inequality (21) holds for V_α introduced in Eq. (23). Now verifying whether it is possible to find V'_α for which (21) holds for arbitrary $|\chi'_\alpha\rangle$ is valuable.

Our measurable bounds are related to the $ALB_\alpha(\rho)$ rather than the concurrence itself, as we have seen in expressions (22) and (39). So we can use (14) to get the relations (31) and (41). Inequality (31)(Inequality (41)) has this advantage that we can omit the summation over those α for which $tr\,(\rho\otimes\rho V_\alpha)\leq 0$ ($tr\,(\rho W_{\sigma\alpha})\geq 0$). This useful property can help us to achieve better results in detecting the entanglement. As an example, W_σ in Eq. (46) is, up to a constant, the summation of three $W_{\sigma\alpha}$. Now, using expression (41), we can omit each $W_{\sigma\alpha}$ for which $tr\,(\rho W_{\sigma\alpha})\geq 0$; But, using W_σ, we can not omit any $W_{\sigma\alpha}$ in Eq. (46). So, as it is shown in Fig. 2, the ability of W_σ in detecting the entanglement reduces more rapidly than the three distinct $W_{\sigma\alpha}$.

Bounds obtained from V_α or $W_{\sigma\alpha}$ are always less than or equal to the $ALB_\alpha(\rho)$. In addition, we have shown that these bounds can not detect bound entangled states. So $ALB_\alpha(\rho)>0$ and $N(\rho)>0$ ($N(\rho)$ is the negativity of the system [48]) are two necessary conditions for detection of the entanglement by V_α or $W_{\sigma\alpha}$. However, the ability of these bounds and also comparing them with other observable bounds, especially those introduced in [8, 14], need further studies. For example, in the definition of $W_{\sigma\alpha}$, mixed states σ can be used simply instead of pure states σ since $ALB_\alpha(\sigma)$ is always computable. Studying the above case seems interesting.

At last, in section V, we have generalized our measurable bounds to the multipartite case. The applicability of these bounds also needs further studies.

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Appendix A

In this appendix, we prove inequality (21) for V_α introduced in Eq. (23). We prove it for $V_{(2)\alpha}$; the case of $V_{(1)\alpha}$ can be done analogously.

Any arbitrary $|\psi\rangle$ and $|\varphi\rangle$ can be decomposed in a separable basis of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, like $|i_A\rangle|j_B\rangle$, as:

$$|\psi\rangle = \sum_{ij} \psi_{ij} |i_A j_B\rangle \,,$$
$$|\varphi\rangle = \sum_{ij} \varphi_{ij} |i_A j_B\rangle \,.$$

Now, from Eq. (23), we have:

$$\begin{aligned}
&\langle\psi|\langle\varphi|V_{(2)\alpha}|\psi\rangle|\varphi\rangle = \\
&2\left[-|\psi_{xq}\varphi_{yq}-\psi_{yq}\varphi_{xq}|^2-|\psi_{xp}\varphi_{yp}-\psi_{yp}\varphi_{xp}|^2+AA\right]\,, \\
&AA=-2Re\left(\psi_{xp}\varphi_{yq}\psi^*_{xq}\varphi^*_{yp}\right)-2Re\left(\psi_{yp}\varphi_{xq}\psi^*_{yq}\varphi^*_{xp}\right) \\
&\quad+2Re\left(\psi_{xp}\varphi_{yq}\psi^*_{yq}\varphi^*_{xp}\right)+2Re\left(\psi_{xq}\varphi_{yp}\psi^*_{yp}\varphi^*_{xq}\right)\,. \qquad \text{(A.1)}
\end{aligned}$$

Also for $|\chi_\alpha\rangle = (|xy\rangle - |yx\rangle)_A\,(|pq\rangle - |qp\rangle)_B$ we have:

$$\begin{aligned}
&|\langle\chi_\alpha|\psi\rangle|\psi\rangle||\langle\chi_\alpha|\varphi\rangle|\varphi\rangle| \\
&=4|\left(\psi_{xp}\psi_{yq}-\psi_{xq}\psi_{yp}\right)\left(\varphi_{xp}\varphi_{yq}-\varphi_{xq}\varphi_{yp}\right)| \\
&\equiv 4|BB|\,. \qquad \text{(A.2)}
\end{aligned}$$

To get the inequality (21), we must show:

$$\begin{aligned}
AA \le 2|BB| &+ |\psi_{xq}\varphi_{yq}-\psi_{yq}\varphi_{xq}|^2 \\
&+|\psi_{xp}\varphi_{yp}-\psi_{yp}\varphi_{xp}|^2\,. \qquad \text{(A.3)}
\end{aligned}$$

If we have:

$$\begin{aligned}
&AA \le 2|BB| + 2|CC|\,, \\
&CC=\left(\psi_{xq}\varphi_{yq}-\psi_{yq}\varphi_{xq}\right)\left(\psi_{xp}\varphi_{yp}-\psi_{yp}\varphi_{xp}\right)\,, \qquad \text{(A.4)}
\end{aligned}$$

then inequality (A.3) holds. To get the inequality (A.4), it is sufficient to have:

$$\begin{aligned}
&\frac{AA}{2} \le |BB+CC| \\
&=|\left(\psi_{xp}\varphi_{yq}-\psi_{yp}\varphi_{xq}\right)\left(\psi^*_{yq}\varphi^*_{xp}-\psi^*_{xq}\varphi^*_{yp}\right)|\,. \qquad \text{(A.5)}
\end{aligned}$$

But, the above expression holds since for any complex number z, we have $Re(z) \le |z|$, which completes the proof.